3.6 Introduction to Linear Transformation

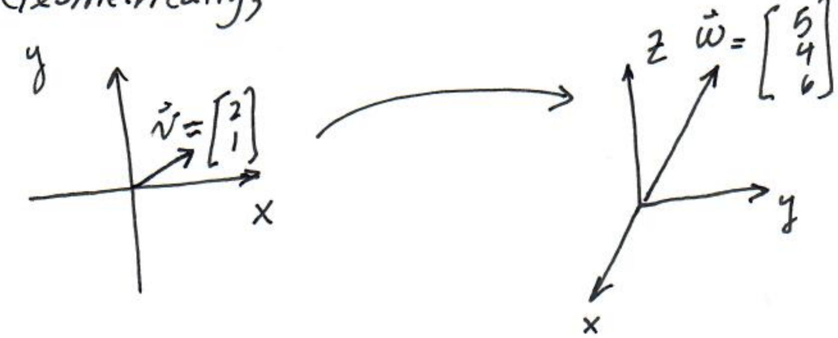
Consider the matrix

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} \text{ and the Vector } \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

then $A\hat{v} = \begin{bmatrix} 5 \\ 4 \\ 6 \end{bmatrix}$

We can see this as a transformation of the vector $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ into $\hat{w} = \begin{bmatrix} 5 \\ 4 \\ 6 \end{bmatrix}$

Geometrically,



We can extend this transformation to all Vectors in $\mathbb{R}^2$ by

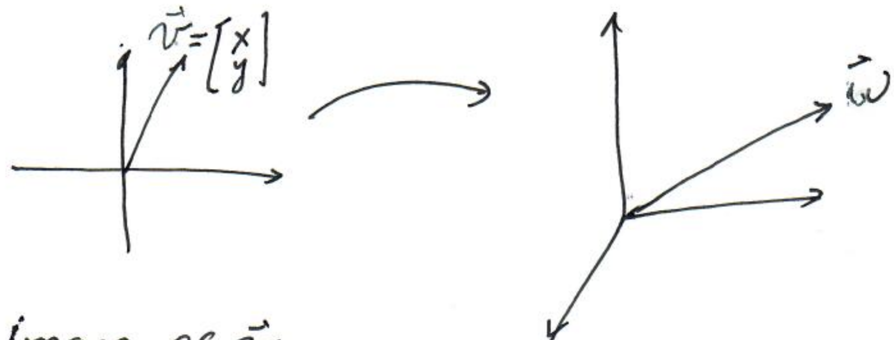
$$\begin{bmatrix} 3 & -1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x - y \\ x + 2y \\ x + 4y \end{bmatrix}$$

and we denote this transformation as

$$T_A\left(\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{=\vec{v}}\right) = \begin{bmatrix} 3x-y \\ x+2y \\ x+4y \end{bmatrix} = \vec{w} \qquad (2.1)$$

Clearly,

$$T_A : \underbrace{\mathbb{R}^2}_{\text{Domain}} \longrightarrow \underbrace{\mathbb{R}^3}_{\text{Codomain}}$$



$T_A(\vec{v})$: image of $\vec{v}$

$\{T_A(\vec{v}) : \vec{v} \in \mathbb{R}^2\}$ = Range of T_A

What is the range of T_A?

$$T_A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x-y \\ x+2y \\ x+4y \end{bmatrix} = \begin{bmatrix} 3x \\ x \\ x \end{bmatrix} + \begin{bmatrix} -y \\ 2y \\ 4y \end{bmatrix} = x\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$$

So, this means that the range of $T_A = Col(A)$.

or all possible linear combinations of the column vectors of A.

This is a plane through the origin,

$$\mathcal{P}: \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ in } \mathbb{R}^3 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} \right\} \text{ in } \mathbb{R}^3.$$

So range of $T_A = \mathcal{P}$. in $\mathbb{R}^3$.

We want to distinguish this type of transformations as T_A from other transformations like

$$T_{NL}: \mathbb{R}^2 \to \mathbb{R}^2$$
$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} xy \\ x^2+y^2 \end{bmatrix}$$

<u>Definition</u>

A transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is called a linear transformation if

1) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$
2) $T(c\vec{v}) = cT(\vec{v})$.

Now, we will check that this is verify for the transformation T_A but not for T_{NL}

$$T_A(\vec{u}+\vec{v}) = T_A\left(\overset{\vec{u}}{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}} + \overset{\vec{v}}{\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}}\right) =$$

$$= T_A\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}\right) = \begin{bmatrix} 3(x_1+x_2) - (y_1+y_2) \\ x_1 + x_2 + 2y_1 + 2y_2 \\ x_1 + x_2 + 4y_1 + 4y_2 \end{bmatrix}$$

$$\overset{\substack{\text{Algebra} \\ \text{in} \\ \mathbb{R}}}{=} \begin{bmatrix} (3x_1 - y_1) + (3x_2 - y_2) \\ (x_1 + 2y_1) + (x_2 + 2y_2) \\ (x_1 + 4y_1) + (x_2 + 4y_2) \end{bmatrix}$$

$$= \begin{bmatrix} 3x_1 - y_1 \\ x_1 + 2y_1 \\ x_1 + 4y_1 \end{bmatrix} + \begin{bmatrix} 3x_2 - y_2 \\ x_2 + 2y_2 \\ x_2 + 4y_2 \end{bmatrix} = T_A\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + T_A\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)$$

$$= T_A(\vec{u}) + T_A(\vec{v}).$$

It's also possible to show

$$T_A(c\vec{v}) = cT_A(\vec{v})$$ in a similar way.

This implies that T_A is a Linear Transformation

We can easily prove that

$$T_{NL}(\vec{u}+\vec{v}) \neq T_{NL}(\vec{u}) + T_{NL}(\vec{v})$$

Therefore T_{NL} is not a Linear transformation.

Alternative for definition of L.T. $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

T is linear if

$$\boxed{T(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2)} \qquad (5.1)$$

for all $\vec{v}_1$ and $\vec{v}_2$ in $\mathbb{R}^n$ and c_1 and c_2 Scalars in $\mathbb{R}$.

Notice that

$$T_A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x-y \\ x+2y \\ x+4y \end{bmatrix} \qquad (5.2)$$

Was obtained from the matrix $A = \begin{bmatrix} 3 & -1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}$.

We can reverse the procedure used and obtain the matrix A from (5.2).

In fact,

$$T_A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x-y \\ x+2y \\ x+4y \end{bmatrix} = \begin{bmatrix} 3x \\ x \\ x \end{bmatrix} + \begin{bmatrix} -y \\ 2y \\ 4y \end{bmatrix} = x\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}.$$

Also, notice that $T \overset{\hat{e}_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ and $T \overset{\hat{e}_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$

Therefore,

$$T_A\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}3 & -1\\ 1 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}T(\hat{e}_1) & T(\hat{e}_2)\\ \downarrow & \downarrow\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

This last calculations suggest that any linear transformation is also a matrix transformation.

<u>Thm</u> <u>3.31</u> $T: \mathbb{R}^n \to \mathbb{R}^m$

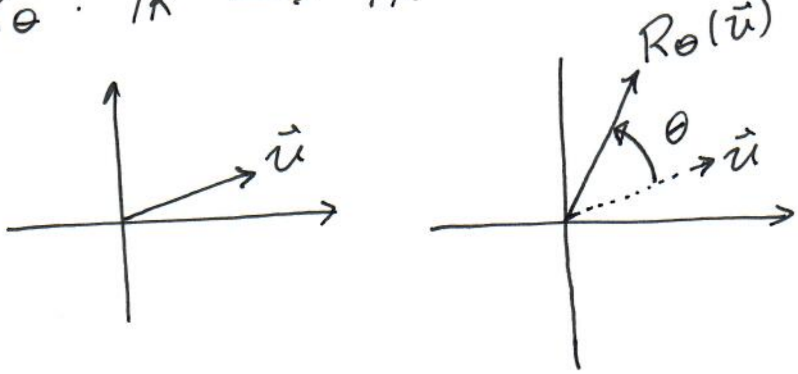
If T is a linear transformation, then T is a matrix transformation, where $A_{m\times n}$ and $A=\begin{bmatrix}T(\hat{e}_1) & \cdots & T(\hat{e}_n)\\ \downarrow & & \end{bmatrix}$.

<u>Proof.</u> (easy).

Rotation about the origin

Transformation

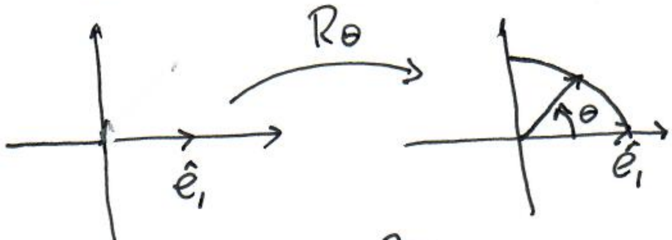
$$R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$$



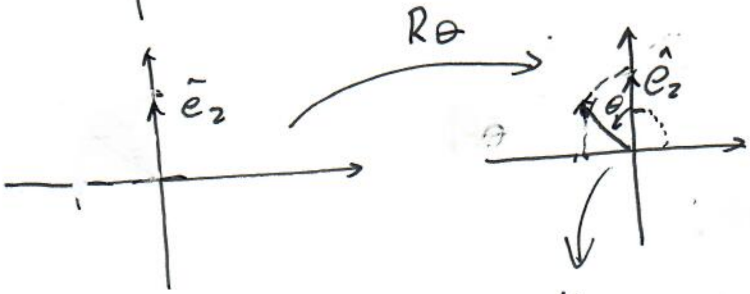
is a linear transformation.

Geometric discussion →

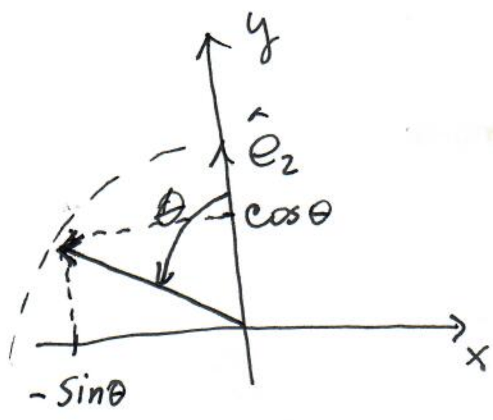
Also,



$$R_\theta(\hat{e}_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$



$$R_\theta(\hat{e}_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$



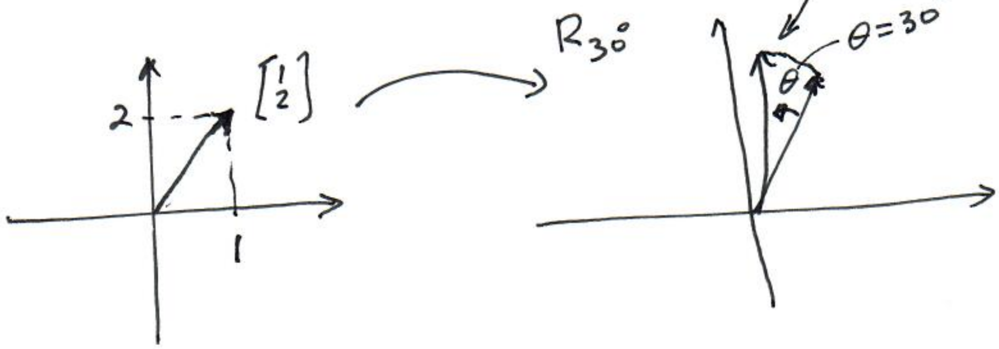
So

$$A_{R_\theta} = \left[R_\theta(\hat{e}_1) \;\; R_\theta(\hat{e}_2) \right] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

If $\theta = 30^\circ = \pi/6$

then $$R_{30^\circ} = \begin{bmatrix} \cos \pi/6 & -\sin \pi/6 \\ \sin \pi/6 & \cos \pi/6 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

Find $$R_{30^\circ}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 - 1 \\ 1/2 + \sqrt{3} \end{bmatrix}$$



– <u>Example 3.59</u> projection interesting.

– Comp. of L.T. $\longrightarrow$ $\underset{[S]\;[T]}{S \circ T} = [S][T]$.

– Inv. of L.T $\longrightarrow$ $[T^{-1}] = [T]^{-1}$.

Composition of Linear Transformation

Consider

$$T : \mathbb{R}^n \to \mathbb{R}^m$$
$$S : \mathbb{R}^m \to \mathbb{R}^p.$$

Two linear transformations. Then, we can define a new linear transf.

$$S \circ T : \mathbb{R}^n \to \mathbb{R}^p$$
$$\vec{v} \to \vec{z}$$

where $\quad S \circ T(\vec{v}) = S(T(\vec{v})) = \vec{z}$

$$\begin{array}{ccccc} \mathbb{R}^n & \xrightarrow{T} & \mathbb{R}^m & \xrightarrow{S} & \mathbb{R}^p \\ \vec{v} & \longrightarrow & T(\vec{v}) & \longrightarrow & S(T(\vec{v})) \end{array}$$

A requisite for the composition of S and T to be well defined is that

Codomain of T = Domain of S.

Thm 3.32

If T and S (as defined above) are linear transformations, then $S \circ T$ is also linear and its associated or standard matrix Satisfies

$$[S \circ T] = [S][T].$$

Inverse of a Linear Transformation

Def.- If $S: \mathbb{R}^n \to \mathbb{R}^n$
$T: \mathbb{R}^n \to \mathbb{R}^n$

and $S \circ T = T \circ S = I$

where $I: \mathbb{R}^n \to \mathbb{R}^n$
$\vec{v} \to \vec{v}$.

We will say than S and T are invertible

and $S^{-1} = T$ and $T^{-1} = S$.

Thm 3.33 If $T: \mathbb{R}^n \to \mathbb{R}^n$ is invertible

then $[T^{-1}] = [T]^{-1}$.